Strictification and Gray categories

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Abstract

Mac Lane's strictification theorem states that every (weak) monoidal category is equivalent to a strict one. This fact extends to 2-categories, but fails in higher dimension: not every 3-category is equivalent to a strict one. In this context, Gray categories provide a compromise: sufficiently strict to be manageable, while remaining sufficiently general to capture every weak 3-category. In this short note, we discuss strictification in category theory, how it relates to coherence and when it fails, and finally informally introduce Gray categories.

In category theory, a categorical structure most often comes equipped with some sort of *structural morphisms* witnessing its properties; for instance, a monoidal structure on a category comes equipped with an associator and unitors, witnessing the associativity and unitality of the monoidal structure. In general, we say that the structure is *strict* if all structural morphisms are identities; in constrast, the generic case is sometimes called *weak*. If every weak structure is suitably equivalent to a strict one, we say that the structure can be *strictified*. This is the case for monoidal categories:

Theorem (Mac Lane's strictification theorem)**.** *Every (weak) monoidal category is monoidally equivalent to a strict monoidal category.*

This result extends to 2-categories: every (weak) 2-category (or *bicategory*) is 2-equivalent to a strict 2-category. However, this stops to hold for 3-categories: there exist (weak) 3-categories (or *tricategories*) that are not 3-equivalent to strict 3-categories.

Whenever the strictification theorem doesn't hold, one can still look for weaker versions where one only strictifies *some* of the structural morphisms. A categorical structure where some of the structural morphisms are strict is called *semi-strict*. Note that in general, there are multiple notions of being semi-strict for a given structure. The goal is then to find a semi-strict structure that preserve *expressiveness* (every weak structure is suitably equivalent to a semi-strict one), while achieving *effectiveness* (sufficiently strict to be manageable in practice).

For 3-categories, one such semi-strict compromise is given by the notion of *Gray categories*. The companion strictification theorem was shown by Gordon, Power and Street in 1995 [3]:[1]

Theorem (strictification theorem for Gray categories)**.** *Every (weak) 3-category is 3-equivalent to a Gray category.*

This note starts in §1 with explaining the close connection between strictification and *coherence*, illustrated in the context of monoidal category. We then introduce the notion of *braided monoidal categories* in §2, one of the simplest cases where full strictification cannot be achieved. Finally, §3 informally introduces Gray categories, emphasizing the analogy with braided monoidal categories.[2]

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[1] See also the detailed account of Gurski [4].

[2] These slides and the paper [9] gave some inspirations for this note, although we cannot pinpoint a specific spot in the text.

§1 Strictification and coherence in monoidal categories

Recall the definition of a monoidal category:

Definition ([1, Definition 6.1.1], nLab)**.** *A* monoidal category *is a category $\mathcal{K}$ which admits a functor $\otimes\colon \mathcal{K} \times \mathcal{K} \to \mathcal{K}$, called* tensor product*, such that:*

(i) There exists a natural isomorphism $\alpha_{x,y,z}\colon (x \otimes y) \otimes z \to x \otimes (y \otimes z)$ called the associator*, satisfying the following naturality condition:*

$$\begin{tikzcd}
((x \otimes y) \otimes z) \otimes w \arrow[rr, "\alpha_{x,y,z} \otimes \mathrm{id}_w"] \arrow[d, "\alpha_{x \otimes y, z, x}"'] & & (x \otimes (y \otimes z)) \otimes w \arrow[d, "\alpha_{x, y \otimes z, w}"] \\
(x \otimes y) \otimes (z \otimes w) \arrow[dr, "\alpha_{x,y,z \otimes w}"'] & & x \otimes ((y \otimes z) \otimes w) \arrow[dl, "\mathrm{id}_x \otimes \alpha_{y,z,w}"] \\
& x \otimes (y \otimes (z \otimes w)) &
\end{tikzcd}$$

(ii) There exists an object $I \in \mathcal{K}$, called the unit*, such that there exists a natural isomorphism $\lambda_x\colon I \otimes x \to x$ and $\rho_x\colon x \otimes I \to x$ called* unitors*, satisfying the following naturality condition:*

$$\begin{tikzcd}
x \otimes (I \otimes y) \arrow[rr, "\alpha_{x,I,y}"] \arrow[dr, "\mathrm{id}_x \otimes \lambda_y"'] & & (x \otimes I) \otimes y \arrow[dl, "\rho_x \otimes \mathrm{id}_y"] \\
& x \otimes y &
\end{tikzcd}$$

A monoidal functor *is a functor $\mathcal{F}\colon (\mathcal{K}, \otimes_{\mathcal{K}}, I_{\mathcal{K}}) \to (\mathcal{L}, \otimes_{\mathcal{L}}, I_{\mathcal{L}})$ between monoidal categories, together with a natural isomorphism $\mu_{x,y}\colon \mathcal{F}(x) \otimes_{\mathcal{L}} \mathcal{F}(y) \to \mathcal{F}(x \otimes_{\mathcal{K}} y)$ satisfying some naturality conditions (see e.g. the nLab).*

Mac Lane's coherence theorem (nLab) for monoidal categories informally states that the naturality conditions are enough to ensure that "every diagram made of α, λ and ρ in $\mathcal{K}$ commutes". Formulated in this way, the statement is not technically correct; indeed, one can construct an example where for some object x, we have $(x \otimes x) \otimes x = x \otimes (x \otimes x)$ while the associator $\alpha_{x,x,x}$ is non-trivial.[3] The correct statement is only concerned with diagrams that "respect the choice of bracketing", which are called *formal diagrams*:

(i) *Mac Lane's coherence theorem (first version):* In a monoidal category $(\mathcal{K}; \alpha, \lambda, \rho)$, every formal diagram made of $\{\alpha^{\pm 1}, \lambda^{\pm 1}, \rho^{\pm 1}\}$ commutes.

Another way to understand formal diagrams is to say that they are exactly "the diagrams made of α, λ and ρ that make sense in any monoidal category". To formalize that point, denote by $\mathcal{Brack}$ the free monoidal category generated by one element $(-)$. The objects of $\mathcal{Brack}$ are all the ways of bracketing a tuple, with abritrary units I thrown in. For instance,

$$(-) \otimes (((-) \otimes (-)) \otimes I)$$

[3]This is Isbell's example, mentioned here and also in [2, remark 2.8.7]. This is closely related to the fact that while a monoidal category is monoidally equivalent to a skeletal monoidal category, it is not in general equivalent to a skeletal strict monoidal category. See also [5, prop. 1.35].

is an object of $\mathcal{B}rack$. Morphisms are formal compositions of $\{\alpha^{\pm 1}, \lambda^{\pm 1}, \rho^{\pm 1}\}$, affecting the bracketing in the obvious way. This category decomposes into connected components $\mathcal{B}rack(n)$, depending on the length n of the tuple (in the above example, the tuple is of length three). If $\mathcal{F}\colon \mathcal{B}rack \to \mathcal{K}$ is a monoidal functor, then the image $\mathcal{F}(D)$ of a diagram $D \subset \mathcal{B}rack$ is a formal diagram, and every formal diagram arises in this way. Mac Lane's coherence theorem can then be reformulated as follows:

(ii) *Mac Lane's coherence theorem (second version):* The connected components $\mathcal{B}rack(n)$ are contractible groupoids.

In other words, there is exactly one morphism between each pair of objects in $\mathcal{B}rack(n)$; equivalently, the underlying graph is a complete graph (discarding identities). This point of view leads the way to yet another version of Mac Lane's coherence theorem, namely Mac Lane's strictification theorem, already mentioned in the introduction:

(iii) *Mac Lane's coherence theorem (third version)*, or *Mac Lane's strictification theorem:* Every (weak) monoidal category is monoidally equivalent to a strict monoidal category.

In general, coherence and strictification are always two sides of the same coin. To illustrate that fact in the context of monoidal categories, we show that (iii) follows from (ii).[4] [5]

Proof that (ii) ⇒ (iii). Let $(\mathcal{K}, \otimes_{\mathcal{K}}, I_{\mathcal{K}})$ be a monoidal category. How should its strictification $\mathcal{K}^{\text{str}}$ be defined? On objects, it follows the naive idea that, if associativity were strict in $\mathcal{K}$, we could juxtapose objects without worrying about brackets:

- objects in $\mathcal{K}^{\text{str}}$ are tuples $\overline{x} = (x_1, x_2, \ldots, x_n)$ of objects in $\mathcal{K}$. The monoidal structure on objects is given by concatenation.

To define morphisms, we think of $\overline{x}$ as representing all the possible bracketings of x_1, x_2 and x_n. More precisely, each such tuple defines a functor

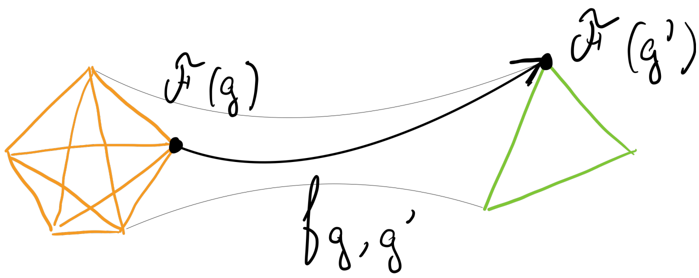
$$\mathrm{ev}_{\overline{x}}\colon \mathcal{B}rack(n) \to \mathcal{K}$$

which "evaluates" a bracketing on $\overline{x}$. For instance:

$$\mathrm{ev}_{\overline{x}}\colon (-) \otimes (((-) \otimes (-)) \otimes I) \mapsto x_1 \otimes ((x_2 \otimes x_3) \otimes I_{\mathcal{K}}).$$

It follows from (ii) that $\mathrm{ev}_{\overline{x}}$ is a *clique*, that is, a functor whose domain is a contractible monoid. Cliques formalize the notion of contractible sub-categories in a given category.

If $\mathcal{F}\colon \mathcal{G} \to \mathcal{K}$ and $\mathcal{F}'\colon \mathcal{G}' \to \mathcal{K}$ are two cliques, a *clique morphism* betwen $\mathcal{F}$ and $\mathcal{F}'$ is a family of morphism $\mathcal{F}(g) \to \mathcal{F}(g')$ for objects $(g, g') \in \mathcal{G} \times \mathcal{G}'$, such that all relevant triangles commute. This family is in fact uniquely determined by a single morphism $\mathcal{F}(g) \to \mathcal{F}(g')$; this is a consequence of the contractibility of $\mathcal{G}$ and $\mathcal{G}'$. Here is a schematic:



[4]The direction (iii) ⇒ (i) is relatively clear: this essentially follows from the fact that formal diagrams commute in strict monoidal categories.

[5]This proof is sketched in this nLab article.

We can now conclude the definition of $\mathcal{K}^{\mathrm{str}}$:

- morphisms in $\mathcal{K}^{\mathrm{str}}$ between tuples $\overline{x}$ and $\overline{y}$ are the clique morphisms between $\mathrm{ev}_{\overline{x}}$ and $\mathrm{ev}_{\overline{y}}$. The monoidal structure in induced from $\otimes_{\mathcal{K}}$.

More precisely, if $f\colon \mathcal{B}rack(\overline{x}) \to \mathcal{B}rack(\overline{y})$ and $f'\colon \mathcal{B}rack(\overline{x}') \to \mathcal{B}rack(\overline{y}')$ are morphisms between bracketings representing clique morphisms $[f]$ and $[f']$, then $[f] \otimes [f'] := [f \otimes f']$. Given that cliques capture all bracketings at the same time, it is straightforward that $[(f \otimes f') \otimes f''] = [f \otimes (f' \otimes f'')]$, and thus the tensor product is strictly associative. One similarly check that it is strictly unital.

Finally, the embedding (and not quotient![6]) $\iota\colon \mathcal{K} \hookrightarrow \mathcal{K}^{\mathrm{str}}$ is given by $x \mapsto (x)$ on objects, with (x) a length-one tuple, and by $f \mapsto [f]$ on morphisms. This embedding is monoidal, with the isomorphism $\iota(x \otimes_{\mathcal{K}} y) \cong \iota(x) \otimes_{\mathcal{K}^{\mathrm{str}}} \iota(y)$ induced by the identity on $x \otimes_{\mathcal{K}} y$. This also shows that ι is essentially surjective. Finally, uniqueness of the clique represented by f shows that this is fully faithful. □

Remark. One streamlined proof of (iii) is analogous to Cayley's theorem, embedding the weak monoidal category in a strict monoidal category of endofunctors (this can also be interpreted as a 2-dimensional Yoneda embedding); this approach was first given in [7], and it is the one presented in [2, 5]. Another proof can be found in [8, p. XI.5].

§2 Braided monoidal categories and the failure of strictification

Definition (nLab)**.** *A* braided monoidal category *is a monoidal category* $(\mathcal{K}, \otimes_{\mathcal{K}}, I_{\mathcal{K}})$ *together with a natural isomorphism*

$$\beta_{x,y}\colon x \otimes y \to y \otimes x,$$

satisfying some further naturality conditions. This isomorphism is called the braiding*.*

A *braided monoidal functor* is a monoidal functor between braided monoidal categories, satisfying some compatibility condition with respect to the braiding.

In the literature, one will find that Mac Lane's strictification theorem extends to the braided setting:

Theorem (Mac Lane's strictification theorem for braided monoidal categories)**.** *Every braided monoidal category is braided monoidally equivalent to a strict one.*

However, the terminology is misleading, as a *strict braided monoidal category* is just a braided monoidal category whose underlying monoidal category is strict. In other words, in a strict braided monoidal category, the braiding may not be strict! Hence, this theorem should rather be understood as a *semi*-strictification theorem. If one forces the braiding to be strict, it forces the equality $x \otimes y = y \otimes x$; in that case, one speaks of a *commutative monoidal category.*[7]

Why doesn't strictification hold in the braided setting? Because coherence doesn't hold. For instance, one does not expect $\beta_{x,x}\colon x \otimes x \to x \otimes x$ to be trivial! However, one can still find a (restricted) coherence theorem for braided monoidal categories:[8]

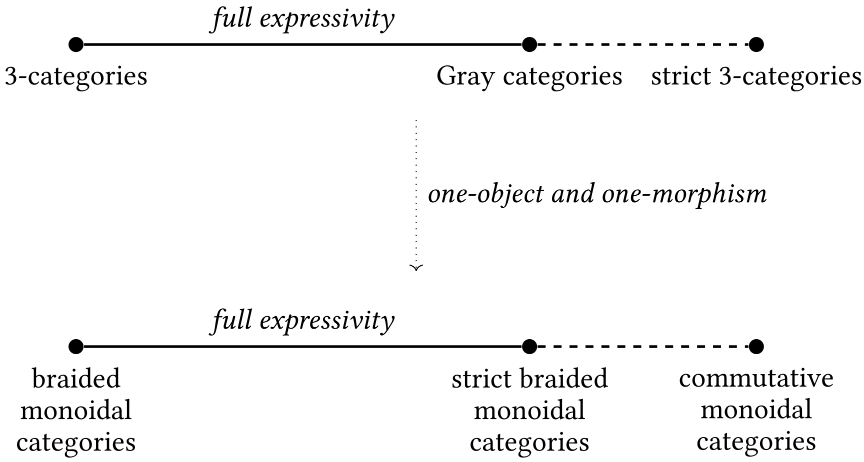
Theorem. *In a braided monoidal category, formal diagrams made up of* $\{\alpha^{-1}, \lambda^{-1}, \rho^{-1}, \beta^{-1}\}$ *commute if they represent element in the braid group.*

[6] See this discussion.
[7] This terminology appears in this paper; see also this MO question.
[8] See for instance [6].

Remark. A braided monoidal category is called *symmetric* if the braiding further satisfies the condition that $\beta_{y,x} \circ \beta_{x,y} = \mathrm{id}_{x \otimes y}$. There are similar results in this setting, namely that every symmetric monoidal category is braided monoidally equivalent to strict symmetric monoidal category (where "strict" again only means that the underlying monoidal category is strict), and formal diagrams made up of $\{\alpha^{\pm 1}, \lambda^{\pm 1}, \rho^{\pm 1}, \beta^{\pm 1}\}$ commute if they represent the same element in the symmetric group.

§3 Strictifying 3-categories with Gray categories

As mentioned in the introduction, Gray categories are semi-strict replacement for 3-categories. In some sense, they are "the furthest one can strictify" while preserving equivalence with generic 3-categories.[9] The relationship between these three notions (3-categories, Gray categories and strict 3-categories) is expressed in the schematic below:



In fact, if one unwraps the definitions in the one-object and one-morphism case, one precisely recovers the analogous situation for braided monoidal category!

We start by recalling strict 2-categories. In a nutshell, a strict 2-category has an underlying category and additional morphisms between morphisms, called *2-moprhisms* (we also rename morphisms as *1-morphisms*). The latter can be depicted using *globular diagrams*:

$$\begin{tikzcd} B & A \arrow[l, bend right=50, "g"'] \arrow[l, bend left=50, "f"] \arrow[l, phantom, "\Uparrow\alpha"] \end{tikzcd} .$$

2-morphisms can be composed in two distinct ways: either vertically, gluing along 1-morphisms (denoted with $\star_1$), or horizontally, gluing along objects (denoted with $\star_0$, formerly the composition $\circ$ in the underlying category):

$$\begin{tikzcd} B & A \arrow[l, bend right=70, "h"'] \arrow[l, "g"'] \arrow[l, bend left=70, "f"] \arrow[l, phantom, shift right=3, "\Uparrow\beta"] \arrow[l, phantom, shift left=3, "\Uparrow\alpha"] \end{tikzcd} = \begin{tikzcd} B & A \arrow[l, bend right=50, "h"'] \arrow[l, bend left=50, "f"] \arrow[l, phantom, "\Uparrow\beta\star_1\alpha"] \end{tikzcd} \qquad \begin{tikzcd} C & B \arrow[l, bend right=50, "g'"'] \arrow[l, bend left=50, "f'"] \arrow[l, phantom, "\Uparrow\beta"] & A \arrow[l, bend right=50, "g"'] \arrow[l, bend left=50, "f"] \arrow[l, phantom, "\Uparrow\alpha"] \end{tikzcd} = \begin{tikzcd} C & A \arrow[l, bend right=50, "g'\circ g"'] \arrow[l, bend left=50, "f'\circ f"] \arrow[l, phantom, "\Uparrow\beta\star_0\alpha"] \end{tikzcd} .$$

[9]Gray categories are not the only direction to strictify a 3-categories. For instance, one can strictify the interchange law but leave units weak. See e.g. the nLab.

Since the 2-category is strict, both compositions are associative and unital. However, we have an extra naturality condition, which captures the compatibility between the two compositions. It is called the *interchange law*:

$$(f \star_0 h) \star_1 (g \star_0 k) = (f \star_1 g) \star_0 (h \star_1 k).$$

In other words, the interchange law ensures that "it doesn't matter whether we first compose vertically and then horizontally, or vice-versa".

It follows from the interchange law that

$$(f \star_0 \mathrm{id}) \star_1 (\mathrm{id} \star_0 g) = f \star_0 g = (\mathrm{id} \star_0 g) \star_1 (f \star_0 \mathrm{id}).$$

In other words, we could also *define* horizontal composition of non-trivial 2-morphisms as either side of this equality, and only require horizontal composition with identities to be defined. We then only record the equality

$$(f \star_0 \mathrm{id}) \star_1 (\mathrm{id} \star_0 g) = (\mathrm{id} \star_0 g) \star_1 (f \star_0 \mathrm{id}).$$

The above is often also referred as the *interchange law.* This provides an equivalent definition of a strict 2-category, which will be useful in our definition of Gray categories.

If the strict 2-category only has one object, then "horizontally composing along objects" is an empty condition, and the horizontal composition is just a tensor product. Hence we can write $\star_0$ as $\otimes$ and $\star_1$ as $\circ$, and the interchange law as

$$(f \otimes \mathrm{id}) \circ (\mathrm{id} \otimes g) = (\mathrm{id} \otimes g) \circ (f \otimes \mathrm{id}).$$

The above encodes the naturality of the tensor product. This shows that a one-object strict 2-category is precisely a strict monoidal category, with 1-morphisms corresponding to objects and 2-morphisms to morphisms. If this monoidal category further has a single object, then by the Eckmann-Hitlon argument we get a commutative monoid.

A *strict 3-category* similarly has an underlying 2-category and additional morphisms between the 2-morphisms, called *3-morphisms.* They can be composed in three different ways, gluing along objects (denoted $\star_0$), along 1-morphisms (denoted $\star_1$) or along 2-morphisms (denoted $\star_2$). Each composition is strictly associative and unital. Similarly to the fact that a one-object one-morphism strict 2-category is a commutative monoid, a one-object one-morphism strict 3-category is a commutative monoidal category.

In contrast, a Gray category also has associative and unital compositions, *but* the interchange law only holds up to isomorphism. This isomorphism is witnessed by a 3-morphism called the *interchangor*:

$$\gamma_{f,g} \colon (f \otimes \mathrm{id}) \circ (\mathrm{id} \otimes g) \Rrightarrow (\mathrm{id} \otimes g) \circ (f \otimes \mathrm{id}),$$

satisfying some coherent conditions.

A one-object one-morphism Gray category is precisely a strict braided category, with the following correspondences:

one-object one-morphism Gray categories		*strict braided monoidal categories*
2-morphisms	$\leftrightarrow$	objects
3-morphisms	$\leftrightarrow$	morphisms
interchangor	$\leftrightarrow$	braiding

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